Интегралы

1. Вычислите:

a) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx =_{\{t=\cos x\}} - \int \frac{1}{t} dt = -\ln t + C = -\ln\cos x + C$

b) $\int x^2 e^x dx = \int x^2 de^x = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2\int x de^x = x^2 e^x - 2xe^x + 2\int e^x dx = x^2 e^x - 2xe^x + 2e^x + C$

2. Вычислите:

a) $\int \frac{x\ln x}{\sqrt{1+x^2}} dx = \int \ln x d\sqrt{1+x^2} = \sqrt{1+x^2}\ln x - \int \frac{\sqrt{1+x^2}}{x} dx =_{\{x=\tan u, dx=\frac{1}{\cos^2 u}du\}} \sqrt{1+x^2}\ln x - \int \frac{1}{\sin u\cdot\cos^2 u} du$

$\int \frac{1}{\sin x\cos^2 x} dx =_{\{t=\cos x, dt=-\sin x dx\}} \int \frac{1}{t^2(t^2-1)} dt = \int \left(-\frac{1}{t^2} - \frac{1}{2(t+1)} + \frac{1}{2(t-1)}\right) = \frac{1}{t} + \frac{1}{2}\ln\frac{1-t}{1+t} + C$

$t = \cos u = \cos\arctan x = \frac{1}{\sqrt{1+x^2}}$

$\int \frac{x\ln x}{\sqrt{1+x^2}} dx = \ln x\sqrt{1+x^2} - \sqrt{1+x^2} - \frac{1}{2}\ln\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$

b) $I = \int e^{2x}\cos x dx = \frac{1}{2}\int \cos x de^{2x} = \frac{1}{2}\left(e^{2x}\cos x + \int e^{2x}\sin x dx\right) = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}\left(e^{2x}\sin x - \int e^{2x}\cos x dx\right) = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x - \frac{1}{4}I$

$\Rightarrow I = \frac{e^{2x}}{5}(2\cos x + \sin x)$

3. Найдите интегралы:

a) $\int \frac{6x-7}{3x^2-7x+1} dx =_{\{t=3x^2-7x+1\}} = \int \frac{1}{t} dt = \ln\left(3x^2-7x+1\right) + C$

b) $\int \frac{2x+1}{x^2-2x+5} dx = \int \left(\frac{2x-2}{x^2-2x+5} - \frac{3}{x^2-2x+5}\right) = \ln\left(x^2-2x+5\right) - \int \frac{3}{x^2-2x+5} dx$

$\int \frac{1}{x^2-2x+5} =_{\{u=\frac{x-1}{2}\}} \frac{1}{2}\int \frac{1}{1+u^2} du = \frac{1}{2}\arctan\frac{x-1}{2} + C$

4. Найдите интеграл:

Указание: замена $t = x - \frac{1}{x}$

$\int \frac{x^2+1}{x^4+1} dx =_{\{t=x-\frac{1}{x}\}}, dx = dt\frac{x^2}{x^2+1} \int \frac{1}{x^2+x^{-2}} dt = \int \frac{1}{t^2+2} dt = \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} + C$

5. Вычислите:

a) $\int \frac{dx}{\sin x} =_{t=\cos x, dt=-\sin x dx} \int \frac{-dt}{1-t^2} = \frac{1}{2}\ln\frac{1-t}{1+t} + C = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C$

b) $\int \arctan x dx = x\arctan x - \int \frac{x}{1+x^2} dx = x\arctan x - \frac{1}{2}\ln(1+x^2)$