Интегралы

1. Вычислите:

a)

$$\int \frac{1}{\cos x} dx = \left[\begin{array}{c} t = \sin x \\ dt = \cos x dx \end{array} \right] \quad \int \frac{1}{1-t^2} dt = \frac{1}{2} \log\left(\frac{1+\sin x}{1-\sin x}\right) + C$$

b)

$$\int x^5 e^{x^3} dx = \left[\begin{array}{c} t = x^3 \\ dt = 3x^2 dx \end{array} \right] \quad \frac{1}{3}\int t e^t dt = \frac{1}{3} e^{x^3}\left(x^3 - 1\right) + C$$

c)

$$\int x^2\sqrt{1-x^2} dx = \left[\begin{array}{c} x = \sin t \\ dx = \cos t dt \end{array} \right] \quad \int \sin^2 t \cos^2 t dt = \frac{1}{4}\int \sin^2 2t dt =$$

$$\left[\begin{array}{c} u = 2t \\ du = 2dt \end{array} \right] \quad \frac{1}{8}\int \sin^2 u du = \frac{1}{8}\int \frac{1-\cos 2u}{2} du = \frac{1}{8}\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C =$$

$$\frac{1}{8}\left(t - \frac{1}{4}\sin 4t\right) + C = \frac{1}{8}\left(\arcsin x - x\sqrt{1-x^2}\left(1-2x^2\right)\right) + C$$

Примечание:

$$\sin(4t) = 2\sin 2t \cos 2t = 4 \sin t \cos t\left(1 - 2\sin^2 t\right) = \left[\ t = \arcsin x\ \right] \quad 4x\sqrt{1-x^2}\left(1-2x^2\right)$$

2. Вычислите:

a) $\int \frac{x}{(x+1)(x-2)(x-3)} dx = \int \frac{-dx}{12(x+1)} + \int \frac{-2dx}{3(x-2)} + \int \frac{3dx}{4(x-3)} = \ldots$

b) $\int \frac{x^4-2x^3-7x^2+10x+12}{x^3-4x^2+x+6} dx = \int \left(x + 2 + \frac{2x}{(x+1)(x-2)(x-3)}\right) dx = \ldots$

3. Вычислите:

a)

$$\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}} dx$$

$$\sqrt{1+x+x^2} = tx + 1 \Rightarrow 1 + x + x^2 = t^2x^2 + 2tx + 1 \Rightarrow x = \frac{2t-1}{1-t^2}, dx = \frac{2\left(1-t+t^2\right)}{\left(1-t^2\right)^2} dt$$

$$\sqrt{1+x+x^2} = \frac{1-t+t^2}{1-t^2}$$

$$\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}} dx = \int \frac{-2t dt}{1-t^2} = \ln|1-t^2| + C = \ln\left|1 - \left(\frac{\sqrt{1+x+x^2}-1}{x}\right)^2\right| + C$$

b) $\int \frac{12x^3+16x^2+9x+2}{\sqrt{4x^2+4x+2}}dx =$ **TODO**

4. Найдите интеграл

$$\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}dx = \int \frac{\sin 2x}{\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x} = \int \frac{\sin 2x}{1 - \frac{1}{2}\sin^2 2x}dx$$

$$\int \frac{2\sin 2x}{1+\cos^2 2x}dx = \left[\begin{array}{c} t = \cos 2x \\ dt = -2\sin 2x dx \end{array} \right] \quad \int \frac{-dt}{1+t^2} = -\arctan t + C = -\arctan \cos 2x + C$$

Интегралы

1.

$$J_n = \int \cos^n x dx = \int \cos^{n-1} x d\sin x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx =$$

$$\cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx = \cos^{n-1} x \sin x + (n-1)\left(J_{n-2} - J_n\right)$$

$$J_n = \frac{1}{n}\left(\cos^{n-1} x \sin x + (n-1)J_{n-2}\right)$$

2.

$$J_n = \int x^n e^{ax}dx = \frac{1}{a}\int x^n de^{ax} = \frac{1}{a}\left(x^n e^{ax} - \int e^{ax} n x^{n-1}dx\right) = \frac{1}{a}x^n e^{ax} - \frac{n}{a}J_{n-1}$$

3.

$$J_n = \int \frac{1}{\left(x^2+a^2\right)^n}dx = \left[\begin{array}{c} a \cdot t = x \\ a \cdot dt = dx \end{array} \right] \quad \frac{1}{a^{2n-1}}\int \frac{1}{x^2+1} =$$

$$\left[\begin{array}{c} t = \tan u \\ dt = \frac{1}{\cos^2 u}du \end{array} \right] \quad \frac{1}{a^n - 1}\int \cos^{2n-2} u du$$

4. Как посчитать интеграл вида?

$$\int \frac{Ax+B}{\left(ax^2+bx+c\right)^n}dx$$

Сведем линейной заменой к предыдущему интегралу и интегралу вида $\int \frac{x}{\left(x^2+a^2\right)^n}dx$

$$\int \frac{x}{\left(x^2+a^2\right)^n}dx = \left[\begin{array}{c} t = x^2 \\ dt = 2xdx \end{array} \right] \quad \frac{1}{2}\int \frac{1}{\left(t+a^2\right)^n dt} = \left[\ u = t + a^2\ \right] \quad \frac{1}{2}\int \frac{1}{u^n}du$$

5. Посчитайте

$$\int \frac{1}{\sqrt[3]{(2+x)(2-x)^5}}dx$$

Указание: ищите замену вида $t = \left(\frac{ax+b}{cx+d}\right)^p$

TODO

6.

$$\int \frac{1}{3\sin x + 4\cos x + 5}dx = \int \frac{1}{6\sin\frac{x}{2}\cos\frac{x}{2} + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + 5\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)}dx = \textbf{TODO}$$